The Concept of Mean Anisotropy of Signals with Nonzero Mean

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***Abstract*—The paper presents a novel concept of the anisotropy-based analysis for the stochastic sequences with nonzero mean. The formulas for the anisotropy of random vector and mean anisotropy of sequence are obtained. Basic types of shaping filter connections are presented.**

I. Introduction

Nowadays the branch of control theory named control and estimation with information constrains is booming (for example, see [1]). The information constraints are used in two ways. From the one hand, the characteristics of information channels (such as channel capacity) are included in the control system model. From the other hand, some information theory notions (such as entropy, relative entropy, and mutual information) are used to characterize the control performance. We can note some main papers connected with the second way (see [2], [3], [4], [5]). This paper follows the second way of the information theory application in control.

Based on the ideas of I.G. Vladimirov, the theory of stochastic robust control was developed starting from the middle of 1990's ([6], [7], [8], [9]). This theory employs the anisotropy functional as an entropy theoretic measure of deviation of the unknown actual noise distribution from the family of Gaussian white noise laws with scalar covariance matrices. Accordingly, the role of a robust performance index is played by the a-anisotropic norm $|\!|\!|F|\!|\!|_a$ of the system F which is defined as largest ratio of the root mean square value of the output of the system to that of the input, provided that the anisotropy of the input disturbance does not exceed a given nonnegative parameter a. Thus, the input anisotropy level a is the size of the statistical uncertainty, and the a-anisotropic norm of the system $|\!|\!|F|\!|\!|_a$ is the worst-case gain which, in the framework of the disturbance attenuation paradigm, is to be minimized.

An important property of the a-anisotropic norm is that it coincides with the scaled H_2 norm of the system for $a = 0$ and converges to the H_∞ norm as $a \to +\infty$. Therefore, $|\!|\!| \cdot |\!|\!|_a$ is an anisotropy-constrained stochastic version of the system norm, which occupies a unifying intermediate position between the H_2 and H_∞ norms applied as performance criteria in the linear quadratic Gaussian (LQG) and H_∞ control theories.

To synthesize the optimal anisotropic controllers, a so-called worst-case shaping filter is to be found. The input of this filter is fed with the multi-dimensional Gaussian white noise with zero mean and scalar covariance matrixes. An interesting question arises: can we build an anisotropy-based robust control theory for the class of random input signals with the nonzero mean?

The formulas for the mean anisotropy of the output signal of the shaping filter with the input Gaussian signal that have the nonzero mean will be the base for the anisotropy-based robust control theory design for the control systems under the random Gaussian disturbances with the nonzero mean.

This paper is focused on the study of the properties of the mean anisotropy of the random signal with the time-invariant mean.

The paper results help one to trace how the mean anisotropy level (coloration of the signal) changes in any point of the complex linear circuit of the control system. This paper can also be used to support physical and engineering society for a "rational" choice of the anisotropy level while designing the anisotropy-based robust controllers.

II. Background

The anisotropy $\mathbf{A}(W)$ of an m-dimensional random vector W with probability density function (pdf) f is defined as

$$\mathbf{A}(W) = \min_{\lambda > 0} \mathbf{D}(f || p_{m,\lambda}), \tag{1}$$

where

$$\mathbf{D}(f || p_{m,\lambda}) = \mathbf{E}_f \left[\ln \left(\frac{f}{p_{m,\lambda}} \right) \right] \tag{2}$$

is the Kullback-Leibler divergence of f from the Gaussian pdf

$$p_{m,\lambda}(x) = (2\pi\lambda)^{-m/2} \exp\left\{ -\frac{x^{\mathrm{T}} x}{2\lambda} \right\},$$

and $\mathbf{E}_f[\,\cdot\,]$ denotes the mathematical expectation in sense of f. Thus, one can interpret the anisotropy $\mathbf{A}(W)$ as the measure of closeness of the certain vector W to the set of Gaussian vectors with pdf $p_{m,\lambda}$, parameterized by λ.

If the vector W is also Gaussian and its pdf has the form

$$f(x) = ((2\pi)^m |S|)^{-1/2} \exp\left\{ -\frac{1}{2} x^{\mathrm{T}} S^{-1} x \right\},$$

where $|\cdot|$ denotes the determinant of matrix, S is positive define matrix, then

$$\mathbf{D}(f || p_{m,\lambda}) = -h(W) - \frac{m}{2} \ln(2\pi\lambda) + \frac{\mathrm{tr} S}{2\lambda}, \tag{3}$$

where $h(W)$ stands for the differential entropy of W defined as

$$h(W) = -\mathbf{E}_f[\ln f],$$

$\mathrm{tr}(\cdot)$ means the trace of a matrix. From (1) and (3) we have got the following formula

$$\mathbf{A}(W) = -\frac{1}{2}\ln\frac{|S|}{\left(\frac{1}{m}\mathrm{tr}S\right)^m}.$$

The standard way to form the colored sequence $\{w_k\}$ is to represent this sequence as an output of the following discrete time-invariant system called the shaping filter:

$$\begin{cases} x_{k+1} = Ax_k + Bv_k, \\ w_k = Cx_k + Dv_k, \end{cases} \quad x_0 = 0, \tag{4}$$

where x_k denotes the filter internal state; the input $\{v_k\}$ is the Gaussian white noise. The mean anisotropy $\overline{\mathbf{A}}(W)$ of the stationary ergodic sequence $W = \{w_k\}$ is defined as the limit

$$\overline{\mathbf{A}}(W) = \lim_{N\to\infty}\frac{\mathbf{A}(W_{0:N-1})}{N}, \quad W_{0:N-1} = \begin{bmatrix} w_0 \\ \dots \\ w_{N-1} \end{bmatrix}. \tag{5}$$

Let Σ be the covariance matrix of the vector $w_k|_{k\to\infty}$, so $\Sigma = \mathrm{cov}(w_k|_{k\to\infty})$. The matrix Σ is connected with the solution $P > 0$ of the Lyapunov equation

$$P = APA^{\mathrm{T}} + BB^{\mathrm{T}}$$

by following expression

$$\Sigma = CPC^{\mathrm{T}} + DD^{\mathrm{T}}.$$

Let Ψ be the covariance matrix of the prediction error vector $\tilde{w}_k|_{k\to\infty} > 0$, $\widetilde{w}_k = w_k - \mathbf{E}[w_k|\{w_j\}_{j<k}]$ as $k \to \infty$.

Define

$$\Xi = \Psi - \Sigma.$$

It is shown in [8] that the expression for the mean anisotropy (5) by virtue of (4) can be written as

$$\overline{\mathbf{A}}(W) = -\frac{1}{2}\ln\frac{|\Sigma + \Xi|}{\left(\frac{1}{m}\mathrm{tr}\Sigma\right)^m}, \tag{6}$$

$$\Xi = CRC^{\mathrm{T}},$$

where R is the solution of the Riccati equation

$$R = ARA^{\mathrm{T}} - \Lambda(\Sigma + \Xi)^{-1}\Lambda^{\mathrm{T}}$$

with the notation

$$\Lambda = BD^{\mathrm{T}} + A(P + R)C^{\mathrm{T}}.$$

These formulas allow to compute the mean anisotropy of the random sequence W generated by the shaping filter (4) in terms of the second-order moments of W on the base of solutions to the respective algebraic Riccati and Lyapunov equations.

III. Main result

Let us consider the Gaussian random vector W with the nonzero mean value μ and covariance matrix S, i.e. with the pdf

$$f(W) = \left((2\pi)^m|S|\right)^{-1/2}\exp\left\{-\frac{1}{2}(W-\mu)^{\mathrm{T}}S^{-1}(W-\mu)\right\}.$$

According to (2), its relative entropy takes the form

$$\mathbf{D}(f||p_{m,\lambda}) = -h(W) - \frac{m}{2}\ln(2\pi\lambda) + \frac{\mathrm{tr}S + \|\mu\|_2^2}{2\lambda},$$

where the expectation μ appeares in contrast to (3), and therefore the anisotropy of the vector W is written as

$$\mathbf{A}(W) = -\frac{1}{2}\ln\frac{|S|}{\left(\frac{1}{m}\left(\mathrm{tr}S + \|\mu\|_2^2\right)\right)^m}. \tag{7}$$

To generate a sequence $\{w_k\}$ of Gaussian random vectors with the nonzero mean value μ, it is required to rewrite (4) in the form

$$\begin{cases} x_{k+1} = Ax_k + Bv_k, \\ w'_k = Cx_k + Dv_k + \mu, \end{cases} \quad x_0 = 0, \tag{8}$$

where as before x_k denotes the filter internal state, and the input sequence $\{v_k\}$ is the Gaussian white noise.

Application of the mean anisotropy definition (5) to the sequence $W' = \{w'_k\}$ gives

$$\overline{\mathbf{A}}(W') = \lim_{N\to\infty}\frac{\mathbf{A}(W'_{0:N-1})}{N},$$

or, equivalently,

$$\overline{\mathbf{A}}(W') = -\frac{1}{2}\ln\frac{\lim\limits_{N\to\infty}|\Sigma'_{0:N-1}|^{1/N}}{\lim\limits_{N\to\infty}\left(\frac{1}{mN}(\mathrm{tr}\Sigma'_{0:N-1} + \|\mu'_{0:N-1}\|_2^2)\right)^m}, \tag{9}$$

where

$$\mu'_{0:N-1} = \begin{bmatrix} \mu \\ \dots \\ \mu \end{bmatrix},$$

and

$$\Sigma'_{0:N-1} = \mathrm{cov}(W'_{0:N-1})$$

stands for the covariance matrix of the vector $W'_{0:N-1}$, which is equal to the covariance matrix of the vector $W_{0:N-1}$, i.e. $\Sigma'_{0:N-1} = \Sigma_{0:N-1}$. It is easy to see that

$$\Sigma_{0:N-1} = \begin{bmatrix} DD^{\mathrm{T}} & \dots & D(CA^{N-2}B)^{\mathrm{T}} \\ \vdots & \ddots & \vdots \\ * & \dots & DD^{\mathrm{T}} + \sum\limits_{k=0}^{N-2} CA^kB(CA^kB)^{\mathrm{T}} \end{bmatrix}$$

and therefore

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{tr}\Sigma_{0:N-1} = \mathrm{tr}\Sigma,$$

where $\Sigma = \mathrm{cov}(w_k)\,|_{k\to\infty}$.

Comparing (6) and (9) by virtue of the above formula, one can conclude that the ergodic property of the considered sequence W leads to the existence of the limit

$$\lim_{N\to\infty}|\Sigma_{0:N-1}|^{1/N} = |\Sigma + \Xi|$$

for the case when $\mu = 0$. Such property also extends to the sequence W' because $\Sigma'_{0:N-1} = \Sigma_{0:N-1}$. Then on the strength of

$$\lim_{N\to\infty} \frac{1}{N} \|\mu'_{0:N-1}\|_2^2 = \|\mu\|_2^2,$$

the next formula for the mean anisotropy of W' arises:

$$\overline{\mathbf{A}}(W') = -\frac{1}{2} \ln \frac{|\Sigma + \Xi|}{\left(\frac{1}{m}\left(\mathrm{tr}\Sigma + \|\mu\|_2^2\right)\right)^m}, \qquad (10)$$

Further we will use the notation $\overline{\mathbf{A}}_\mu(W)$ for the mean anisotropy of the sequence W with nonzero mean μ.

Theorem 1: The formula (10) for the mean anisotropy $\overline{\mathbf{A}}_\mu(W)$ of the sequence W generated by the filter

$$G: \begin{cases} x_{k+1} = Ax_k + Bv_k, \\ w_k = Cx_k + Dv_k + \mu, \end{cases}$$

can be written as

$$\overline{\mathbf{A}}_\mu(W) = \overline{\mathbf{A}}_0(W) + \frac{m}{2} \ln \frac{\|G\|_2^2 + \|\mu\|_2^2}{\|G\|_2^2},$$

where $\|G\|_2$ denotes the H_2-norm of the matrix transfer function $G(z) = D + C(zI_m - A)^{-1}B$.

Proof: The straightforward calculations give

$$\begin{aligned} \overline{\mathbf{A}}_\mu(W) &= -\frac{1}{2} \ln \frac{|\Sigma + \Xi|}{\left(\frac{1}{m}(\mathrm{tr}\Sigma + \|\mu\|_2^2)\right)^m} \\ &= -\frac{1}{2} \ln \left(\frac{|\Sigma + \Xi|}{(\frac{1}{m}\mathrm{tr}\Sigma)^m} \frac{(\mathrm{tr}\Sigma)^m}{(\mathrm{tr}\Sigma + \|\mu\|_2^2)^m} \right) \\ &= -\frac{1}{2} \ln \frac{|\Sigma + \Xi|}{(\frac{1}{m}\mathrm{tr}\Sigma)^m} - \frac{1}{2} \ln \left(\frac{\mathrm{tr}\Sigma}{\mathrm{tr}\Sigma + \|\mu\|_2^2} \right)^m \\ &= \overline{\mathbf{A}}_0(W) + \frac{m}{2} \ln \frac{\|G\|_2^2 + \|\mu\|_2^2}{\|G\|_2^2}, \end{aligned}$$

where $\mathrm{tr}\Sigma = \|G\|_2^2$.

EXAMPLE

Consider the sequence W generated from the Gaussian white noise by the shaping filter

$$G \sim \left[\begin{array}{cc|cc} 0.395493 & 0.011521 & 0.33866 & -0.0013469 \\ -0.19875 & 0.52119 & 0.64526 & -0.012195 \\ \hline 0.71817 & -0.080732 & 0.45846 & -0.57262 \\ -0.89001 & 0.09717 & -0.57262 & -0.45846 \end{array} \right].$$

The mean anisotropy $\overline{\mathbf{A}}_0(W) = 0.2$ if the mean value of W equals to zero ($\mu = 0$). According to (10),

$$\overline{\mathbf{A}}_\mu(W) = -\frac{1}{2} \ln \frac{|\Sigma + \Xi|}{\left(\frac{1}{m}\left(\mathrm{tr}\Sigma + \|\mu\|_2^2\right)\right)^m}$$

if $\mu \neq 0$. If

$$\mu = \begin{bmatrix} 0.3 \\ -1 \end{bmatrix},$$

then

$$\Sigma = \begin{bmatrix} 0.695 & 0.12235 \\ 0.12235 & 0.66501 \end{bmatrix}, \Xi = \begin{bmatrix} -0.13826 & -0.12235 \\ -0.12235 & -0.10827 \end{bmatrix},$$

$$\|\mu\|_2 \simeq 1.044, \quad m = 2.$$

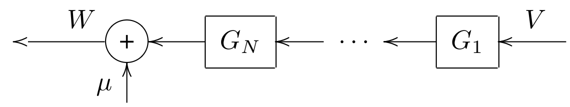

Fig. 1: Series connection of N filters

That is, the mean anisotropy of the sequence $\overline{\mathbf{A}}_\mu(W) \simeq 0.7886$ for the case of the nonzero mean.

One can verify this result with the help of Theorem 1. Since $\|G\|_2 \simeq 1.1662$, then

$$\overline{\mathbf{A}}_0(W) + \frac{m}{2} \ln \frac{\|G\|_2^2 + \|\mu\|_2^2}{\|G\|_2^2} \simeq 0.7886.$$

IV. FILTER CONNECTIONS

Let us consider the most common forms of filter connections such as serial, parallel and multiplex, and derive the formulas of the mean anisotropy for each of them. Further we will use the notation $\overline{\mathbf{A}}_\mu(G)$ for the mean anisotropy of the sequence W formed by the shaping filter G.

A. Serial connection

Let the sequence $\{w_k\}$ be shaped by the filter G, which is represented by the series connection of N filters G_k, $k = \overline{1, N}$, as shown in figure 1.

According to Theorem 1, it is true that

$$\overline{\mathbf{A}}_\mu(G) = \overline{\mathbf{A}}_0(G) + \frac{m}{2} \ln \frac{\|G\|_2^2 + \|\mu\|_2^2}{\|G\|_2^2},$$

where $G = G_N \cdots G_1$. It is shown in [10] that in the case of the series connection of N filters the mean anisotropy $\overline{\mathbf{A}}_0(G)$ satisfies

$$\overline{\mathbf{A}}_0(G) \le \sum_{k=1}^{N} \overline{\mathbf{A}}_0(G_k) + \sum_{k=2}^{N} \frac{m}{2} \ln \frac{m\|G_k\|_\infty^2}{\|G_k\|_2^2}.$$

Therefore,

$$\begin{aligned} \overline{\mathbf{A}}_\mu(G) &= \sum_{k=1}^{N} \overline{\mathbf{A}}_0(G_k) + \sum_{k=2}^{N} \frac{m}{2} \ln \frac{m\|G_k \ldots G_1\|_2^2}{\|G_k\|_2^2 \|G_{k-1} \ldots G_1\|_2^2} \\ &+ \frac{m}{2} \ln \frac{\|G_N \ldots G_1\|_2^2 + \|\mu\|_2^2}{\|G_N \ldots G_1\|_2^2} \\ &\le \sum_{k=1}^{N} \overline{\mathbf{A}}_0(G_k) + \sum_{k=2}^{N} \frac{m}{2} \ln \frac{m\|G_k\|_\infty^2}{\|G_k\|_2^2} \\ &+ \frac{m}{2} \ln \frac{\|G_N \ldots G_1\|_2^2 + \|\mu\|_2^2}{\|G_N \ldots G_1\|_2^2}. \end{aligned}$$

B. Parallel connection

Let the sequence $\{w_k\}$ be shaped by the filter G, which is represented by the parallel connection of N filters G_k, $k = \overline{1, N}$, as shown in figure 2.

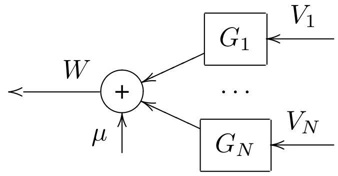

Fig. 2: Parallel connection of N filters

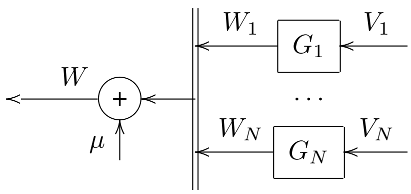

Fig. 3: Multiplex connection of N filters

It is shown in [10] that in the case of the parallel connection of N filters the mean anisotropy $\overline{\mathbf{A}}_0(G)$ is bounded by

$$\begin{aligned}\overline{\mathbf{A}}_0(G) \leq S_0 &= \min\left\{ \sum_{k=1}^{N} \overline{\mathbf{A}}_0(G_k) \frac{\|G_k\|_2^2}{\|G\|_2^2} \; ; \right. \\ &\left. \min_{k=\overline{1,N}} \left(\overline{\mathbf{A}}_0(G_k) + \frac{m}{2} \ln \frac{\|G\|_2^2}{\|G_k\|_2^2} \right) \right\},\end{aligned}$$

where $G = [\; G_1 \; \cdots \; G_N \;]$, and hence $\|G\|_2^2 = \sum\limits_{k=1}^{N} \|G_k\|_2^2$.

Thus, according to Theorem 1,

$$\overline{\mathbf{A}}_\mu(G) \leq S_0 + \frac{m}{2} \ln \frac{\|G\|_2^2 + \|\mu\|_2^2}{\|G\|_2^2}.$$

C. Multiplex connection

Let the sequence $\{w_k\}$ be shaped by the filter G which is represented by the multiplex connection of N filters G_k, $k = \overline{1, N}$, as shown in figure 3, where

$$w_k = \begin{bmatrix} w_k^1 \\ \vdots \\ w_k^N \end{bmatrix} \quad \text{and} \quad \mu = \begin{bmatrix} \mu^1 \\ \vdots \\ \mu^N \end{bmatrix}.$$

It is shown in [10] that in the case of the multiplex connection of N filters the mean anisotropy $\overline{\mathbf{A}}_0(G)$ is written as

$$\overline{\mathbf{A}}_0(G) = \sum_{k=1}^{N} \left(\overline{\mathbf{A}}_0(G_k) + \frac{m_k}{2} \ln \frac{|||G|||_0^2}{|||G_k|||_0^2} \right).$$

Using [11, Lemma 3.21] for the case $\{N = 2, \; m_1 = 1\}$ one can obtain

$$\overline{\mathbf{A}}_\mu(G) = -\frac{1}{2} \ln \frac{|\Sigma_1 + \Xi_1||\Sigma_2 + \Xi_2|}{\left(\frac{1}{m}(\mathrm{tr}\Sigma_1 + \mathrm{tr}\Sigma_2 + \|\mu\|_2^2)\right)^m},$$

where $\mathrm{tr}\Sigma_1 = \Sigma_1 \geq 0$, $\Xi_1 \leq 0$, and therefore

$$\begin{aligned}\min_{G_1} \overline{\mathbf{A}}_\mu(G) &= \overline{\mathbf{A}}_\mu(G)\Big|_{\Xi_1 = 0,\; \Sigma_1 = \frac{\mathrm{tr}\Sigma_2 + \|\mu\|_2^2}{m-1}} \\ &= -\frac{1}{2} \ln \frac{|\Sigma_2 + \Xi_2|}{\left(\frac{1}{m-1}(\mathrm{tr}\Sigma_2 + \|\mu\|_2^2)\right)^{m-1}} \\ &= \overline{\mathbf{A}}_\mu(G_2).\end{aligned}$$

Hence,

$$\overline{\mathbf{A}}_\mu(G) \geq \overline{\mathbf{A}}_\mu(G_2).$$

V. Conclusion

The paper introduces the notion of the mean anisotropy of the Gaussian sequence with nonzero time-invariant mean. The formula for calculation of the mean anisotropy of such the sequence has been obtained. This result is the first step toward the anisotropy-based stochastic robust control theory for the systems fed with the random signals with the nonzero mean value. The relation with the well-known notion of the mean anisotropy [3] and the behavior of the mean anisotropy in some standard connection of the filters has been given.

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